JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 21, No. 6, November–December 1998

Algorithms for Attitude Determination Using the Global Positioning System

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Techniques are discussed for attitude determination using the global positioning system (GPS) differential phase measurements, assuming that the cycle integer ambiguities are known. The problem of attitude determination is posed as a parameter optimization problem. One proposed set of optimal solutions, which includes solutions of Wahba's problem, is based on least-squares fit of some attitude parameters to a set of vector measurements. The use of these algorithms requires the conversion of the basic GPS scalar phase measurements into unit vectors. It is shown that when the GPS antennas constitute two axes of a Cartesian coordinate system, the conversion is immediate. When this is not the case, a more elaborate transformation is required. The necessary conversion formulae for both cases are developed and demonstrated in an example. Another possible approach is based on a least-squares fit of the attitude quaternion to the GPS phase measurements themselves. The cost function of the fit is given in the literature in the most straightforward formulation as a function of the attitude matrix. Conversion is presented of the matrix-based cost function to a quaternion-based cost function that corresponds to the cost function minimized by QUEST. However, unlike the QUEST cost function, the converted cost function is not a simple quadratic form; therefore, the simple QUEST solution is not applicable in this case. An iterative solution for finding the optimal quaternion is derived and demonstrated through numerical examples. The algorithms can handle cases of planar antenna arrays and, thus, cover a deficiency in earlier algorithms.

I. Introduction

ATTITUDE determination using the global positioning system (GPS) carrier signals has been given considerable attention in the last decade.[1–3] Much attention was given to concept, hardware, and algorithm development, as well as to testing. Algorithms for GPS attitude determination given differential phase measurements can be broken into two parts: 1) integer resolution and 2) attitude calculations. Several methods for integer resolution were presented in the literature, e.g., see Refs. 1 and 4. In this work we assume that the integer ambiguity is solved, and we are concerned only with the second part, namely, with attitude calculation.

In this work only two baselines (three antennas) are used, which necessarily form a planar array. It covers a deficiency in the earlier work, where a close initial guess at the attitude is not known and where Cohen's[1] fast method will not work because the antennas form a planar array.

The current implementation calls for three antennas, although the algorithms that will be presented can handle more. Let us designate them as antennas 0, 1, and 2. The three antennas determine a plane to which one can attach a so-called antenna coordinate system.

As an introduction to the more general case that follows, we first discuss a special array where the baselines form the axes of a Cartesian triad. The three antennas are so installed as to form a right angle with antenna 0, the master antenna, at the vertex. The antenna coordinate system is defined as follows. The origin of this system is at antenna 0, the system x axis $\boldsymbol{a}_1$ is along the line connecting antenna 0 and antenna 1, and $\boldsymbol{a}_2$, the y axis of the system, is along the line that connects antenna 0 with antenna 2. The third axis $\boldsymbol{a}_3$ is, of course, normal to the antenna's plane and is directed according to the right-hand rule. Let $\boldsymbol{s}_i$ be a unit vector in the direction of an observed GPS satellite, which is designated as satellite number i; then B_{ji}, the projection of $\boldsymbol{s}_i$ on the axis $\boldsymbol{a}_j$, $j = 1, 2$, is equal to $\cos(\alpha_{ji})$, where α_{ji} is the angle between $\boldsymbol{a}_j$ and $\boldsymbol{s}_i$. This can be written in the antenna coordinate system a as follows:

$$B_{ji} = \cos(\alpha_{ji}) = \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i \tag{1}$$

where T denotes the transpose and D_a^e is the transformation matrix from e, the reference coordinate system in which $\boldsymbol{s}_i$ is resolved, to the antenna system. (It is assumed that the components of $\boldsymbol{s}_i$ in e are known.) The projection B_{ji} is related to the phase difference between simultaneous phase measurements at antenna 0 and antenna j of the carrier broadcasted by the GPS satellite i. This phase difference φ_{ji} is given in carrier wavelength and can be written as follows:

$$\varphi_{ji} = \varphi'_{ji} + N_{ji} \tag{2}$$

where φ'_{ji} is a fraction of wavelength, N_{ji} is an integer. The phase measurement is performed using carrier phase differencing, which yields φ'_{ji} only, but there are techniques[5,6] for quick determination of N_{ji}. Therefore, knowing φ'_{ji} is equivalent to knowing φ_{ji}, and it is assumed that φ_{ji} is the measured quantity. It is easy to show that

$$B_{ji} = N_{ji}(\lambda/b_j) + \varphi'_{ji}(\lambda/b_j) = \varphi_{ji}(\lambda/b_j) \tag{3}$$

where b_j is the distance between antenna 0 and antenna j, and λ is the wavelength. The phase difference is a measured value, which is used in attitude determination, that is, in the computation of D_a^e. The measurement of φ_{ji} introduces measurement errors in B_{ji} that stem from structure flexure, baseline uncertainties, electronic delays, etc. The measurement error is e_{ji}. As a result of the measurement errors,

Received April 7, 1997; presented as Paper 97-3616 at the AIAA Guidance, Navigation, and Control Conference, New Orleans, LA, Aug. 11–13, 1997; revision received April 14, 1998; accepted for publication April 23, 1998.

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the following equation, rather than Eq. (3), relates the phase to the attitude:

$$B_{ji} = \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i + e_{ji} \tag{4}$$

hence,

$$e_{ji}^2 = \left| B_{ji} - \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i \right|^2 \tag{5}$$

This leads to the definition of the cost function

$$\rho\left(D_a^e\right) = \frac{1}{4}\sum_i^n p_i \sum_j^2 e_{ji}^2 \tag{6}$$

where n is the number of satellites in track and p_i is a weight given to the measurement error associated with each of the n satellites. The least-squares fit of the attitude matrix to the measurements is that D_a^e that minimizes the cost function. Note that the introduction of the coefficient $\frac{1}{4}$ (or any other coefficient) in the cost function does not have any influence on the resulting fit. It was done here for ease in later developments. Using Eqs. (5) and (6) the cost function becomes

$$\rho\left(D_a^e\right) = \frac{1}{4}\sum_i^n p_i \sum_j^2 \left| B_{ji} - \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_j \right|^2 \tag{7}$$

The purpose of this work is to find algorithms that yield the attitude matrix (or other equivalent attitude parameterizations) that minimizes ρ.

II. Attitude Determination Using GPS Vectorized Observations

Several efficient algorithms for attitude determination based on a least-squares fit of the attitude to vector measurements were introduced in the past. To make use of these algorithms, the phase measurements have to be converted into vector measurements. Distinction is made between the case where the antennas physically define a Cartesian coordinate system and the case where they do not. The two cases are discussed next.

A. Physically Defined Cartesian Coordinate System

If, as mentioned earlier, the antennas are so installed as to form a Cartesian coordinate system, then the conversion is quite simple. The unit vector $\boldsymbol{s}_i$ to satellite i is expressed in the antenna a coordinate system as follows:

$$\boldsymbol{s}_{ia} = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \left(1 - B_{1i}^2 - B_{2i}^2\right)^{\frac{1}{2}} \end{bmatrix} \tag{8}$$

Note that the third component of $\boldsymbol{s}_{ia}$ is chosen to be the positive root of the expression in parentheses. This was done because only the signals of those GPS satellites that are above the antenna plane, and thus in the positive direction of the $\boldsymbol{a}_3$ axis, are received by the antennas. The vector $\boldsymbol{s}_{ia}$, resolved in Earth reference coordinates, is denoted by $\boldsymbol{s}_{ie}$. The latter is easily computed because both the satellite and the vehicle position are known in Earth coordinates. With the pairs $\boldsymbol{s}_{ia}$, $\boldsymbol{s}_{ie}$ on hand, $i = 1, 2, \ldots, n$, one can replace Eq. (7) by the following cost function introduced by Wahba[7]:

$$\rho'\left(D_a^e\right) = \frac{1}{4}\sum_i^n p_i \left| \boldsymbol{s}_{ia} - D_a^e \boldsymbol{s}_{ie} \right|^2 \tag{9}$$

and use QUEST,[8] FOAM,[9] or REQUEST[10] to obtain a weighted least-squares attitude quaternion (matrix, when FOAM is used) fit that minimizes ρ'. (In fact, if one is interested in a good, albeit not optimal, attitude matrix, one can even use the TRIAD[8,11] or the optimized TRIAD[12] algorithms.) For the sake of comparison between QUEST, which operates on measured vectors, and an algorithm that will be developed later, which operates on phase measurements, a short description of QUEST is given next.

Because D_a^e is a known function of the attitude quaternion[13] $\boldsymbol{q}$, then $\rho(D_a^e)$ can be replaced by $w(\boldsymbol{q})$, where

$$w(\boldsymbol{q}) = \frac{1}{4}\sum_i^n p_i \left| \boldsymbol{s}_{ia} - D_a^e(q) \boldsymbol{s}_{ie} \right|^2 \tag{10}$$

It can be shown that $\boldsymbol{q}^*$, the $\boldsymbol{q}$ that minimizes $w(\boldsymbol{q})$, is the same $\boldsymbol{q}$ that maximizes the cost function

$$\zeta(\boldsymbol{q}) = \tfrac{1}{2}\boldsymbol{q}^T K \boldsymbol{q} \tag{11}$$

where

$$K = \left[\begin{array}{c|c} S - \sigma I & z \\ \hline z^T & \sigma \end{array}\right] \tag{12}$$

and where

$$m_n = \sum_{i=1}^n p_i \tag{13a}$$

$$\sigma = \frac{1}{m_n}\sum_{i=1}^n p_i \boldsymbol{s}_{ia}^T \boldsymbol{s}_{ie} \tag{13b}$$

$$B = \frac{1}{m_n}\sum_{i=1}^n p_i \boldsymbol{s}_{ia} \boldsymbol{s}_{ie}^T \tag{13c}$$

$$S = B + B^T \tag{13d}$$

$$z = \frac{1}{m_n}\sum_{i=1}^n p_i (\boldsymbol{s}_{ia} \times \boldsymbol{s}_{ie}) \tag{13e}$$

The matrix I is the third-order identity matrix. It turns out that $\boldsymbol{q}^*$ is the eigenvector that corresponds to the largest eigenvalue of K. QUEST[8] is an algorithm that yields this $\boldsymbol{q}^*$.

Because the transformation from the antenna a coordinate system to the body b system is known, one may compute the final attitude matrix that transforms vectors from the body to the Earth coordinate system. Let the known transformation matrix from a to b be T_b^a, then

$$D_b^e = T_b^a D_a^e \tag{14}$$

B. Arbitrarily Placed Antennas

If the antenna arrangement does not constitute a Cartesian coordinate system, one can still convert the phase measurements into vector measurements. Figure 1 presents such a case, where the three antennas still (and always) define a plane; the line connecting antenna 0 with antenna 2 is still in the $\boldsymbol{a}_1\boldsymbol{a}_2$ plane, but it does not coincide with the $\boldsymbol{a}_2$ axis. The definition of the antenna coordinate system a is obvious. The angle between $\boldsymbol{c}$, a unit vector along the line connecting the number 0 and the number 2 antennas, and the $\boldsymbol{a}_1$ axis is β, which is a known angle. Following the notations of the Introduction section, denote the projection of $\boldsymbol{s}_i$ on $\boldsymbol{c}$ by B_{2i}, whereas the projection on the $\boldsymbol{a}_1$ axis is B_{1i}. To convert this information to

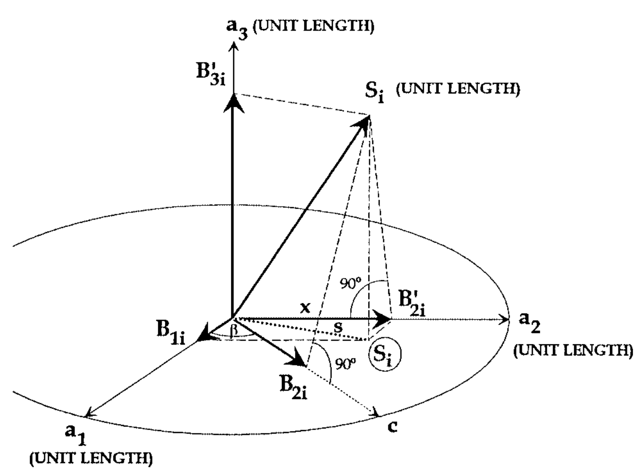

Fig. 1 Definition of the antenna coordinate system.

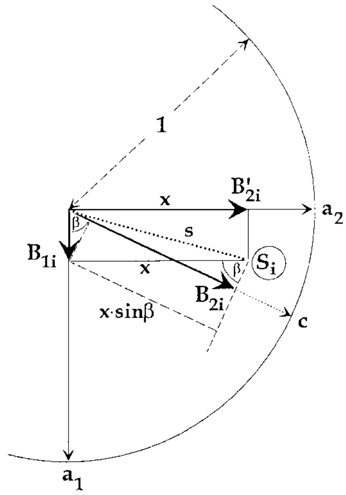

Fig. 2 Computation of B'_{2i}.

a vector observation, one needs to know B'_{2i}, the projection of $\boldsymbol{s}_i$ on the $\boldsymbol{a}_2$ axis. Figure 2 shows the geometric relations between B_{1i}, B_{2i}, and B'_{2i}. Figure 2 is a top view of the $\boldsymbol{a}_1\boldsymbol{a}_2$ plane, where $\boldsymbol{s}$ is the projection of $\boldsymbol{s}_i$ on this plane. Obviously,

$$x \cdot \sin\beta = B_{2i} - B_{1i}\cos\beta \tag{15}$$

solving for x and realizing that $B'_{2i} = x$, one obtains

$$B'_{2i} = (1/\sin\beta)(B_{2i} - B_{1i}\cos\beta) \tag{16}$$

Using Eq. (16), Eq. (8) can be written as

$$\boldsymbol{s}_{ia} = \begin{bmatrix} B_{1i} \\ (1/\sin\beta)(B_{2i} - B_{1i}\cos\beta) \\ \left[1 - B_{1i}^2 - (1/\sin\beta)(B_{2i} - B_{1i}\cos\beta)^2\right]^{\frac{1}{2}} \end{bmatrix} \tag{17}$$

Once the $\boldsymbol{s}_{ia}$ vectors have been computed using Eq. (17) for each available satellite i, one is back at the situation handled earlier (see Sec. II.A) and can use one of the known weighted least-squares fitting algorithms to find the attitude matrix D_a^e, and then the known T_b^a matrix to compute D_b^e, as given in Eq. (14).

III. Attitude Determination Using GPS Phase Measurements Directly

A. Cost Function Conversion to Quadratic Forms

Recall Eq. (7), where now the reference coordinate system is the Earth e system

$$\rho\left(D_a^e\right) = \frac{1}{4}\sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i\right|^2 \tag{18}$$

We wish to find D_a^e that minimizes $\rho(D_a^e)$. Because, as mentioned earlier, D_a^e is a known function of the attitude quarternion $\boldsymbol{q}$, then $\rho(D_a^e)$ can be replaced by $J(\boldsymbol{q})$, where

$$J(\boldsymbol{q}) = \frac{1}{4}\sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{a}_j^T D_a^e(\boldsymbol{q})\boldsymbol{s}_i\right|^2 \tag{19}$$

To facilitate the search for the quarternion $\boldsymbol{q}^*$ that minimizes $J(\boldsymbol{q})$, the latter is now converted into a function of matrix quadratic forms. Realizing that $\boldsymbol{a}_j^T D_a^e(\boldsymbol{q})\boldsymbol{s}_i$ is the computed value of B_{ji}, denote the former by B_c. Also, for ease of notations, in the following development we drop the subscripts j, a, i, and the superscript e. Thus, B_c is expressed as follows:

$$B_c = \boldsymbol{a}^T D(\boldsymbol{q})\boldsymbol{s} \tag{20}$$

The transformation $D(\boldsymbol{q})$ of $\boldsymbol{s}$ can be expressed in quaternion terms as (see Appendix D of Ref. 13)

$$D(\boldsymbol{q})\boldsymbol{s} = \tilde{q} \otimes \tilde{s} \otimes \tilde{q}^{-1} \tag{21}$$

where the tilde denotes the expression of a quaternion as an extension of the complex number; i.e.,

$$\tilde{q} = iq_1 + jq_2 + kq_3 + q_4 \tag{22}$$

$$\tilde{q}^{-1} = -iq_1 - jq_2 - kq_3 + q_4 \tag{23}$$

$$\tilde{s} = is_x + js_y + ks_z + 0 \tag{24}$$

$\otimes$ denotes quaternion multiplication, and s_x, s_y, and s_z are the components of s. Note that if the notation $\boldsymbol{uv}$ is used to denote the column matrix whose entries are the four components of the quaternion product $\tilde{u} \otimes \tilde{v}$ of any $\tilde{u}$ and $\tilde{v}$ quaternions, then

$$\boldsymbol{uv} = \begin{bmatrix} u_4 & u_3 & -u_2 & u_1 \\ -u_3 & u_4 & u_1 & u_2 \\ u_2 & -u_1 & u_4 & u_3 \\ -u_1 & -u_2 & -u_3 & u_4 \end{bmatrix} \boldsymbol{v} = \boldsymbol{Uv} \tag{25}$$

In view of Eq. (21) and using the rule of Eq. (25), the following can be written:

$$D(\boldsymbol{q})\boldsymbol{s} = QS\boldsymbol{q}^{-1} \tag{26}$$

where

$$Q = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \tag{27a}$$

$$S = \begin{bmatrix} 0 & s_z & -s_y & s_x \\ -s_z & 0 & s_x & s_y \\ s_y & -s_x & 0 & s_z \\ -s_x & -s_y & -s_z & 0 \end{bmatrix} \tag{27b}$$

$$\boldsymbol{q}^{-1} = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix} \tag{27c}$$

therefore,

$$\boldsymbol{a}^T D(\boldsymbol{q})s = \boldsymbol{a}^T QS\boldsymbol{q}^{-1} \tag{28}$$

but $\boldsymbol{a}^T Q$ can be written as

$$\boldsymbol{a}^T Q = \boldsymbol{q}^T A \tag{29}$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y & a_x \\ a_z & 0 & -a_x & a_y \\ -a_y & a_x & 0 & a_z \\ a_x & a_y & a_z & 0 \end{bmatrix} \tag{30}$$

therefore, from Eqs. (20), (28), and (29) the following is obtained:

$$B_c = \boldsymbol{q}^T AS\boldsymbol{q}^{-1} \tag{31}$$

It can be easily seen that

$$AS = \left[\begin{array}{c|c} [\boldsymbol{a}\times] & \boldsymbol{a} \\ \hline \boldsymbol{a}^T & 0 \end{array}\right] \left[\begin{array}{c|c} -[\boldsymbol{s}\times] & \boldsymbol{s} \\ \hline -\boldsymbol{s}^T & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{matrix} -[\boldsymbol{a}\times][\boldsymbol{s}\times] \\ -\boldsymbol{a}\boldsymbol{s}^T \end{matrix} & [\boldsymbol{a}\times]\boldsymbol{s} \\ \hline -\boldsymbol{a}^T[\boldsymbol{s}\times] & \boldsymbol{a}^T\boldsymbol{s} \end{array}\right] \tag{32}$$

where $[\boldsymbol{a}\times]$ and $[\boldsymbol{s}\times]$ are the cross-product matrices of $\boldsymbol{a}$ and $\boldsymbol{s}$, respectively. A straightforward development reveals that

$$-[\boldsymbol{a}\times][\boldsymbol{s}\times] = I(\boldsymbol{a}^T\boldsymbol{s}) - \boldsymbol{s}\boldsymbol{a}^T \tag{33a}$$

and

$$-\boldsymbol{a}^T[\boldsymbol{s}\times] = -[\boldsymbol{a}\times\boldsymbol{s}]^T \tag{33b}$$

Therefore, Eq. (32) can be written as

$$AS = \left[\begin{array}{c|c} \begin{array}{c} I(\boldsymbol{a}^T\boldsymbol{s}) \\ -(\boldsymbol{a}\boldsymbol{s}^T + \boldsymbol{s}\boldsymbol{a}^T) \end{array} & \boldsymbol{a}\times\boldsymbol{s} \\ \hline -[\boldsymbol{a}\times\boldsymbol{s}]^T & \boldsymbol{a}^T\boldsymbol{s} \end{array}\right] \tag{34}$$

Define

$$C = \boldsymbol{a}\boldsymbol{s}^T \tag{35a}$$

$$E = C + C^T \tag{35b}$$

$$\boldsymbol{p} = \boldsymbol{a}\times\boldsymbol{s} \tag{35c}$$

$$\mu = \boldsymbol{a}^T\boldsymbol{s} \tag{35d}$$

then

$$AS = \left[\begin{array}{c|c} \mu I - E & \boldsymbol{p} \\ \hline -\boldsymbol{p}^T & \mu \end{array}\right] \tag{36}$$

Substitution of Eq. (36) into Eq. (31) yields

$$B_c = \boldsymbol{q}^T \left[\begin{array}{c|c} \mu I - E & \boldsymbol{p} \\ \hline -\boldsymbol{p}^T & \mu \end{array}\right] \boldsymbol{q}^{-1} \tag{37}$$

Let

$$\boldsymbol{e}^T = [q_1, q_2, q_3] \tag{38a}$$

then one can write

$$\boldsymbol{q} = \left[\begin{array}{c} \boldsymbol{e} \\ \hline q_4 \end{array}\right] \tag{38b}$$

$$\boldsymbol{q}^{-1} = \left[\begin{array}{c} -\boldsymbol{e} \\ \hline q_4 \end{array}\right] \tag{38c}$$

thus, Eq. (37) can be written as

$$\begin{aligned} B_c &= \boldsymbol{q}^T \left[\begin{array}{c|c} \mu I - E & \boldsymbol{p} \\ \hline -\boldsymbol{p}^T & \mu \end{array}\right] \boldsymbol{q}^{-1} = \boldsymbol{q}^T \left[\begin{array}{c|c} \mu I - E & \boldsymbol{p} \\ \hline -\boldsymbol{p}^T & \mu \end{array}\right] \left[\begin{array}{c} -\boldsymbol{e} \\ \hline q_4 \end{array}\right] \\ &= \boldsymbol{q}^T \left[\begin{array}{c|c} E - \mu I & \boldsymbol{p} \\ \hline \boldsymbol{p}^T & \mu \end{array}\right] \left[\begin{array}{c} \boldsymbol{e} \\ \hline q_4 \end{array}\right] = \boldsymbol{q}^T \left[\begin{array}{c|c} E - \mu I & \boldsymbol{p} \\ \hline \boldsymbol{p}^T & \mu \end{array}\right] \boldsymbol{q} \end{aligned} \tag{39}$$

Define

$$L = \left[\begin{array}{c|c} E - \mu I & \boldsymbol{p} \\ \hline \boldsymbol{p}^T & \mu \end{array}\right] \tag{40}$$

then Eq. (39) can be written as

$$B_c = \boldsymbol{q}^T L \boldsymbol{q} \tag{41}$$

When considering the ith satellite and the jth antenna direction, the last equation becomes

$$B_{c,ji} = \boldsymbol{a}_j^T D_a^e(\boldsymbol{q})\boldsymbol{s}_i = \boldsymbol{q}^T L_{ji}\boldsymbol{q} \tag{42}$$

Substitution of Eq. (42) into Eq. (19) yields

$$J(\boldsymbol{q}) = \frac{1}{4}\sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{q}^T L_{ji}\boldsymbol{q}\right|^2 \tag{43}$$

Define

$$\Phi_{ji} = B_{ji} I \tag{44}$$

then because $\boldsymbol{q}^T\boldsymbol{q} = 1$, one can write

$$B_{ji} = \boldsymbol{q}^T \Phi_{ji}\boldsymbol{q} \tag{45}$$

therefore,

$$B_{ji} - \boldsymbol{q}^T L_{ji}\boldsymbol{q} = \boldsymbol{q}^T[\Phi_{ji} - L_{ji}]\boldsymbol{q} \tag{46}$$

Let

$$M_{ji} = p_i(\Phi_{ji} - L_{ji}) \tag{47}$$

then using Eqs. (46) and (47) in Eq. (43), the following is obtained:

$$J(\boldsymbol{q}) = \frac{1}{4}\sum_i^n \sum_j^2 \left|\boldsymbol{q}^T M_{ji}\boldsymbol{q}\right|^2 \tag{48}$$

or

$$J(\boldsymbol{q}) = \frac{1}{4}\boldsymbol{q}^T \left[\sum_i^n \sum_j^2 M_{ji}\boldsymbol{q}\boldsymbol{q}^T M_{ji}\right]\boldsymbol{q} \tag{49}$$

B. Finding the Optimal $\boldsymbol{q}$

The problem of finding the matrix D_a^e that minimizes $\rho(D_a^e)$ defined in Eq. (18) or, equivalently, finding $\boldsymbol{q}$ that minimizes $J(\boldsymbol{q})$ of Eq. (19) has been transformed into finding $\boldsymbol{q}$ that minimizes $J(\boldsymbol{q})$ of either Eq. (48) or Eq. (49). Unfortunately $J(\boldsymbol{q})$, is quartic in $\boldsymbol{q}$, whereas the cost function that has to be optimized when solving Wahba's[7] problem is only quadratic in $\boldsymbol{q}$. For this reason the QUEST solution is not suitable in the present case. One needs to use some other methods for minimizing $J(\boldsymbol{q})$. An iterative solution is suggested here that is based on the gradient projection technique.[14]

Consider the cost function of Eq. (49), where $\boldsymbol{q}$ is subject to the constraint

$$g(\boldsymbol{q}) \equiv \boldsymbol{q}^T\boldsymbol{q} - 1 = 0 \tag{50}$$

Suppose that, as a result of the iterative technique to reduce J, $\boldsymbol{q}_k$ was computed at the kth iteration. Perform now the $k+1$st iteration by changing $\boldsymbol{q}$ as follows:

$$\boldsymbol{q}_{k+1} = \boldsymbol{q}_k + \varepsilon\boldsymbol{h} \tag{51}$$

where $\boldsymbol{h}$ is a four-element column matrix, which determines the direction one moves from $\boldsymbol{q}_k$ to $\boldsymbol{q}_{k+1}$ in $\mathbb{R}^4$, and ε is the distance one moves in this direction. Substituting Eq. (51) into Eq. (49) yields the following cost function at $\boldsymbol{q}_{k+1}$:

$$\begin{aligned} J(\varepsilon, \boldsymbol{h}) &= \frac{1}{4}(\boldsymbol{q}_k + \varepsilon\boldsymbol{h})^T \\ &\times \left[\sum_i^n \sum_j^2 M_{ji}(\boldsymbol{q}_k + \varepsilon\boldsymbol{h})(\boldsymbol{q}_k + \varepsilon\boldsymbol{h})^T M_{ji}\right](\boldsymbol{q}_k + \varepsilon\boldsymbol{h}) \end{aligned} \tag{52}$$

For a given direction $\boldsymbol{h}$, one wishes to move at a distance ε that will minimize J; that is, one wants to move from $J_k = J(\boldsymbol{q}_k)$ to $J_{k+1} = J(\boldsymbol{q}_{k+1})$ at the steepest descent route. The rate of descent of J from point $\boldsymbol{q}_k$ to point $\boldsymbol{q}_{k+1}$ is the partial derivative of J with respect to ε at the point $\varepsilon = 0$. When performing this operation on J, given in Eq. (52), the following is obtained:

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \boldsymbol{q}_k^T \left[\sum_i^n \sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}\right]\boldsymbol{h} \tag{53}$$

Note that the last equation yields the rate of descent but not necessarily the steepest one. While moving from $\boldsymbol{q}_k$ to $\boldsymbol{q}_{k+1}$ the constraint of Eq. (50) still has to be satisfied. Satisfying the constraint implies that when moving an incremental distance, g has to stay zero. In other words, the rate of change of g has to be zero at the point $\boldsymbol{q}_k$. Put in analytic terms and using Eqs. (50) and (51), it means that

$$\left.\frac{\mathrm{d}g}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[(\boldsymbol{q}_k + \varepsilon\boldsymbol{h})^T(\boldsymbol{q}_k + \varepsilon\boldsymbol{h}) - 1\Big]\right|_{\varepsilon=0} = 0 \tag{54}$$

which yields

$$\boldsymbol{q}_k^T\boldsymbol{h} = 0 \tag{55}$$

As for $\boldsymbol{h}$, because it is a direction, it is actually a unit vector; therefore, its length has to be 1, i.e.,

$$\boldsymbol{h}^T\boldsymbol{h} - 1 = 0 \tag{56}$$

Everything is ready now for the computation of the direction of the steepest descent. The direction of the steepest descent is determined by $\boldsymbol{h}$. The column matrix $\boldsymbol{h}$ is sought that yields the smallest (the most negative) value of the derivative expressed by Eq. (53). Define this derivative as $\Psi(\boldsymbol{h})$, i.e.,

$$\Psi(\boldsymbol{h}) = \boldsymbol{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}\right]\boldsymbol{h} \tag{57}$$

which is the function to be minimized with respect to $\boldsymbol{h}$ subject to the constraints expressed by Eqs. (55) and (56). As usual, this is being done by adding the constraints to $\Psi(\boldsymbol{h})$ using the scalar Lagrange multipliers λ_1 and λ_2 and minimizing the new function. Thus, the new function $\eta(\boldsymbol{h})$ is defined as follows:

$$\eta(\boldsymbol{h}) = \boldsymbol{q}_k^T\left(\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}\right)\boldsymbol{h} + \lambda_1\boldsymbol{q}_k^T\boldsymbol{h} + \lambda_2(\boldsymbol{h}^T\boldsymbol{h} - 1) \tag{58}$$

This is the function that is to be minimized with respect to $\boldsymbol{h}$. The concept of directional derivative is used to accomplish that. Accordingly, the direction of the steepest descent is $\boldsymbol{h}_0$, then any other direction $\boldsymbol{h}$ can be expressed as

$$\boldsymbol{h} = \boldsymbol{h}_0 + \nu\boldsymbol{d} \tag{59}$$

where $\boldsymbol{d}$ is the direction from $\boldsymbol{h}_0$ to $\boldsymbol{h}$ and ν is the distance one has to move in this direction to reach $\boldsymbol{h}$ starting at $\boldsymbol{h}_0$. Substitution of the latter expression for $\boldsymbol{h}$ into Eq. (58) yields

$$\eta(\boldsymbol{h}_0 + \nu\boldsymbol{d}) = \boldsymbol{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} + \lambda_1 I\right](\boldsymbol{h}_0 + \nu\boldsymbol{d}) + \lambda_2\Big[(\boldsymbol{h}_0 + \nu\boldsymbol{d})^T(\boldsymbol{h}_0 + \nu\boldsymbol{d}) - 1\Big] \tag{60}$$

A necessary condition for $\boldsymbol{h}_0$ to be a stationary point is

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}\nu}\right|_{\nu=0} = 0 \qquad \forall\boldsymbol{d} \tag{61}$$

Applying this condition to $\eta(\boldsymbol{h}_0 + \nu\boldsymbol{d})$ of Eq. (60) yields

$$\left\{\boldsymbol{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} + \lambda_1 I\right] + 2\lambda_2\boldsymbol{h}_0^T\right\}\boldsymbol{d} = 0 \qquad \forall\boldsymbol{d} \tag{62}$$

This condition can hold only if

$$\boldsymbol{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} + \lambda_1 I\right] + 2\lambda_2\boldsymbol{h}_0^T = 0 \tag{63}$$

which yields

$$\boldsymbol{h}_0 = -\frac{1}{2\lambda_2}\left[\sum_j^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} + \lambda_1 I\right]\boldsymbol{q}_k \tag{64}$$

Because $\boldsymbol{h}_0$ has to satisfy the condition of Eq. (55), then from the last equation

$$-\boldsymbol{q}_k^T\frac{1}{2\lambda_2}\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} + \lambda_1 I\right]\boldsymbol{q}_k = 0 \tag{65}$$

which yields

$$\frac{1}{2\lambda_2}\left[\sum_i^n\sum_j^2 \boldsymbol{q}_k^T M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}\boldsymbol{q}_k + \lambda_1\boldsymbol{q}_k^T\boldsymbol{q}_k\right] = 0 \tag{66}$$

Because $\boldsymbol{q}_k^T\boldsymbol{q}_k = 1$, the last equation implies that

$$\lambda_1 = -\sum_i^n\sum_j^2 \boldsymbol{q}_k^T M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}\boldsymbol{q}_k \tag{67}$$

A comparison between the last equation and Eq. (49) indicates that

$$\lambda_1 = -4J(\boldsymbol{q}_k) \tag{68}$$

Substitution of this result into Eq. (64) yields

$$\boldsymbol{h}_0 = -\frac{1}{2\lambda_2}\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} - 4J(\boldsymbol{q}_k)I\right]\boldsymbol{q}_k \tag{69}$$

To find λ_2, substitute the last expression for $\boldsymbol{h}_0$ into the constraint expressed in Eq. (56) to obtain

$$\left(\frac{1}{2\lambda_2}\right)^2\boldsymbol{q}_k^T\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} - 4J(\boldsymbol{q}_k)I\right] \times\left[\sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} - 4J(\boldsymbol{q}_k)I\right]\boldsymbol{q}_k = 1 \tag{70}$$

Define

$$C_k = \sum_i^n\sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji} \tag{71}$$

then using this definition, Eq. (70) can be written as

$$\lambda_2^2 = \tfrac{1}{4}\boldsymbol{q}_k^T[C_k - 4J(\boldsymbol{q}_k)I][C_k - 4J(\boldsymbol{q}_k)I]\boldsymbol{q}_k \tag{72}$$

which yields

$$\lambda_2^2 = \tfrac{1}{4}\boldsymbol{q}_k^T C_k C_k\boldsymbol{q}_k - 2\boldsymbol{q}_k^T J(\boldsymbol{q}_k)C_k\boldsymbol{q}_k + 4\,\boldsymbol{q}_k^T J(\boldsymbol{q}_k)^2\boldsymbol{q}_k \tag{73}$$

The last equation can be written as

$$\lambda_2^2 = \tfrac{1}{4}\boldsymbol{q}_k^T C_k C_k\boldsymbol{q}_k - 2J(\boldsymbol{q}_k)\boldsymbol{q}_k^T C_k\boldsymbol{q}_k + 4J(\boldsymbol{q}_k)^2\boldsymbol{q}_k^T\boldsymbol{q}_k \tag{74}$$

which, after noting that $\boldsymbol{q}_k^T C_k\boldsymbol{q}_k = 4J(\boldsymbol{q}_k)$, can be written as

$$\lambda_2^2 = \tfrac{1}{4}\boldsymbol{q}_k^T C_k C_k\boldsymbol{q}_k - 8J(\boldsymbol{q}_k)^2 + 4J(\boldsymbol{q}_k)^2 \tag{75}$$

Let

$$\boldsymbol{v}_k = C_k\boldsymbol{q}_k \tag{76}$$

then from Eq. (75) one obtains

$$\lambda_2 = \pm\Big[\tfrac{1}{4}\boldsymbol{v}_k^T\boldsymbol{v}_k - 4J(\boldsymbol{q}_k)^2\Big]^{\frac{1}{2}} \tag{77}$$

To choose the appropriate sign for λ_2, it is necessary to examine the second derivative of $\mu(\boldsymbol{h}_0 + \nu\boldsymbol{d})$ with respect to ν evaluated at $\nu = 0$. Using Eq. (60), it is evident that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}\mu(\boldsymbol{h}_0 + \nu\boldsymbol{d})\right|_{\nu=0} = 2\lambda_2\boldsymbol{d}^T\boldsymbol{d} \tag{78}$$

For $\boldsymbol{h}_0$ to be a minimum point, the second derivative has to be positive; therefore, the positive sign has to be chosen in the computation of λ_2 in Eq. (77). With this in mind and using the definition of C_k, Eq. (69) is written as

$$\boldsymbol{h}_0 = -\frac{1}{2|\lambda_2|}[C_k - 4J(\boldsymbol{q}_k)I]\boldsymbol{q}_k \tag{79}$$

Recall that $\boldsymbol{h}_0$ is the direction of the steepest descent. Substituting the last expression for $\boldsymbol{h}_0$ into Eq. (51) yields

$$\boldsymbol{q}_{k+1} = \left\{I - \frac{\varepsilon}{2|\lambda_2|}[C_k - 4J(\boldsymbol{q}_k)I]\right\}\boldsymbol{q}_k \tag{80}$$

Finally, let

$$W_k = \frac{1}{2|\lambda_2|}[C_k - 4J(\boldsymbol{q}_k)I] \tag{81}$$

then Eq. (80) can be written as

$$\boldsymbol{q}_{k+1} = (I - \varepsilon W_k)\boldsymbol{q}_k \tag{82}$$

A value should be selected for ε. In principle this can be done by substituting $\boldsymbol{q}_{k+1}$ of the last equation into $J(\varepsilon, \boldsymbol{h})$ given in Eq. (52) and then minimizing the result with respect to ε. This, however, yields a complicated third-order equation in ε whose solution has to be obtained at every time step. Another possible approach is of finding empirically a suitable value for ε. In summary the recursive algorithm for minimizing the cost function of Eq. (49) is as follows.

1) Determine $\boldsymbol{q}_1$, the initial guess of $\boldsymbol{q}$, and set $k = 1$.

2) Compute

$$C_k = \sum_i^n \sum_j^2 M_{ji}\boldsymbol{q}_k\boldsymbol{q}_k^T M_{ji}$$

3) Compute $J(\boldsymbol{q}_k) = \boldsymbol{q}_k^T C_k \boldsymbol{q}_k$.

4) Compute $W_k = (1/2|\lambda_2|)[C_k - 4J(\boldsymbol{q}_k)I]$.

5) Compute $\boldsymbol{q}_{k+1} = (I - \varepsilon W_k)\boldsymbol{q}_k$.

6) If $|\boldsymbol{q}_{k+1} - \boldsymbol{q}_k| \le \delta$, where δ is a predetermined constant, then stop. Otherwise increase the argument by 1 and go back to step 2.

C. Example

1. Vectorized Phase Measurements

In this simulated example there are five GPS satellites. The unit vectors to the five satellites $\boldsymbol{s}_i$, $i = 1, 2, \ldots, 5$, are defined as

$$\boldsymbol{s}_1 = \begin{bmatrix} 0.953 \\ 0.095 \\ 0.286 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} -0.195 \\ 0.976 \\ 0.098 \end{bmatrix}, \quad \boldsymbol{s}_3 = \begin{bmatrix} -0.432 \\ -0.259 \\ 0.864 \end{bmatrix}$$

$$\boldsymbol{s}_4 = \begin{bmatrix} -0.316 \\ 0.632 \\ 0.707 \end{bmatrix}, \quad \boldsymbol{s}_5 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

These five directions are chosen randomly from a sample that meets the following constraints. The elevation of the satellites has to be more than 5 deg over the horizon, and the separation between the vectors has to be between 30 and 150 deg. The vectors $\boldsymbol{s}_i$ are expressed in the reference coordinate system. The angle β between the $\boldsymbol{a}_1$ axis and $\boldsymbol{c}$ (see Fig. 2) is 50 deg. Thus

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \cos(50) \\ \sin(50) \\ 0 \end{bmatrix}$$

The transformation matrix D_a^e from the reference to the antenna coordinates and its corresponding quarternion are

$$D_a^e = \begin{bmatrix} 0.713 & 0.659 & 0.241 \\ -0.579 & 0.359 & 0.732 \\ 0.396 & -0.661 & 0.638 \end{bmatrix}$$

$$\boldsymbol{q}^T = [0.423, 0.047, 0.376, 0.823]$$

For this geometry the nominal (error free) phase measurements are

$$B_{11} = 0.811, \quad B_{12} = 0.527, \quad B_{13} = -0.270$$

$$B_{14} = 0.362, \quad B_{15} = 0.931$$

$$B_{21} = -0.309, \quad B_{22} = 0.535, \quad B_{23} = 0.789$$

$$B_{24} = 0.928, \quad B_{25} = 0.295$$

Note that because $\boldsymbol{a}_2 \neq \boldsymbol{c}$, i.e., $\beta \neq 90$ deg, B'_{2i}, $i = 1, 2, \ldots, 5$, are computed using Eq. (16). When vectorizing the phase measurements according to Eq. (17), and using QUEST to compute $\boldsymbol{q}^*$, the achieved accuracy is

$$|\boldsymbol{q} - \boldsymbol{q}^*| < 1 \times 10^{-15}$$

Next, zero-mean random measurement error is added to the GPS phase measurements. The respective standard deviations of the error for each of the five satellites are

$$\sigma_1 = 0.01, \quad \sigma_2 = 0.05, \quad \sigma_3 = 0.03$$

$$\sigma_4 = 0.02, \quad \sigma_5 = 0.02$$

The errors themselves are errors in B_{ji}, that is, in the projection of $\boldsymbol{s}_i$, the unit vector to satellite i, on the antenna coordinate system axis j [see Eq. (1)]. In other words, they are errors in $\varphi_{ji}\lambda/b_j$ [see Eq. (3)], that is, errors in $\cos(\alpha_{ji})$, which are unitless. These errors are due to uncertainty in the value of b_j (the uncertainty in the distance to antenna j), and errors in the measurements of the phase angle φ_{ji} due to structure flexure and electronic noise. When again vectorizing the phase measurements according to Eq. (17), and using QUEST to compute the optimal attitude quaternion $\boldsymbol{q}^*$, the following corresponding errors (in degrees) in yaw, pitch, and roll are obtained:

$$\delta\psi = -0.755, \quad \delta\theta = -0.462, \quad \delta\phi = -0.535$$

2. Direct Phase Measurements

Instead of using the vectorized measurements and consequently the QUEST algorithm, here we find the attitude quaternion using the iterative algorithm that we developed for finding $\boldsymbol{q}^*$ directly from the phase measurements themselves. In other words, we use the iterative algorithm to find the quaternion that minimizes J of Eq. (49). We start the iteration with the arbitrarily chosen initial quaternion

$$\hat{\boldsymbol{q}}^T = [0.451, 0.107, 0.215, 0.859]$$

that corresponds to the following initial attitude estimation error, in degrees, expressed in terms of Euler angles:

$$\delta\psi = 14.05, \quad \delta\theta = -14.78, \quad \delta\phi = 0.802$$

The optimal iteration step size is found empirically to be $\varepsilon = 2.75$. The iterative solution settles on

$$\hat{\boldsymbol{q}}^T = [0.42363, 0.04862, 0.37751, 0.82198]$$

and the final attitude estimation error, in degrees, in terms of yaw, pitch, and roll is

$$\delta\psi = -0.251, \quad \delta\theta = -0.110, \quad \delta\phi = -0.035$$

Observing the convergence rate of the recurrent solutions to the correct one reveals that the convergence rate is only linear; however, the final accuracy of the iterative solution is better than that of the QUEST solution when applied to the vectorized phase measurements. An error analysis reveals that the error associated with the QUEST solution contains a term that does not exist when using the cost function $J(\boldsymbol{q})$ of Eq. (49) to find $\boldsymbol{q}$. The term is a function of the GPS satellite elevations with respect to the antenna coordinates z axis. This term diminishes when the elevations are high. This is the reason why we obtain better results when using the iterative solution that minimizes $J(\boldsymbol{q})$. The two solutions are, of course, identical when they process ideal measurements.

IV. Summary

Algorithms are presented for attitude determination using phase difference between GPS signals arriving at different antennas. Because the number of measurements is greater than the number of the unknown attitude parameters, and because the phase measurements are corrupted by noise, it is advantageous to find the attitude as a least-squares fit. The cost function to be minimized in the fitting process is that given in Eq. (7).

It was shown that when the body-mounted antennas constitute two axes of a so-called antenna coordinate system, the phase measurements can be easily converted into vector measurements. Then, when representing the attitude by the quaternion of rotation, the cost function becomes the one given in Eq. (10), and the least-squares fit can be found using one of the available algorithms such as QUEST and REQUEST.

It is shown that even when the antennas do not constitute two axes of the antenna coordinate system it is still possible to project the phase measurements onto axes of an antenna coordinate system and proceed as before to find a least-squares fit of the attitude parameters. The use of QUEST in the case where the antennas did not constitute two axes of the antenna coordinate system is demonstrated in an example of error free as well as error ridden phase measurements.

Another possible approach is treated, which is based on a least-squares fit of the attitude quaternion to the basic GPS phase measurements. In the literature, the cost function of the attitude fit is given as a function of the attitude matrix, that is, in the form of Eq. (7). It is, however, desirable to express the cost function as a function of the attitude quaternion. This stems from the success attained in quaternion fitting to vector measurements, which is achieved using QUEST. The conversion is presented of the matrix-based cost function to a quaternion-based cost function, which is given in Eqs. (48) and (49). The latter corresponds to Eq. (11), the cost function minimized by QUEST. A comparison between the latter function and that of Eq. (48) reveals that, unlike the case of vector measurements, where the cost function reduces to a quadratic form of a symmetric matrix, in the case of phase measurements the cost function is a sum of squares of quadratic forms; therefore, a simple QUEST-like solution is not applicable in this case.

A possible solution to the problem of finding $\boldsymbol{q}^*$ that minimizes $J(\boldsymbol{q})$ of Eq. (48) is an iterative one. Indeed, such a solution is presented. It is based on the gradient projection technique, which is used to develop a steepest descent search for the local minimum of the cost function. When the search converges to the minimum cost, the iterated quaternion converges to the sought optimal quaternion. The use of the iterative algorithm is demonstrated through a numerical example. It is found that the iteration process converges slowly. It is recommended that other methods, either analytic or iterative, for minimizing $J(\boldsymbol{q})$ be sought. Finally, it should be noted that the algorithms presented in this work cover a deficiency in earlier work in that they are also applicable to attitude determination systems employing planar antenna arrays.

References

[1] Cohen, C. E., "Attitude Determination Using GPS," Ph.D. Thesis, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA, Dec. 1992.

[2] Lightsey, E. G., Cohen, C. E., Feess, W. A., and Parkinson, B. M., "Analysis of Spacecraft Attitude Measurements Using Onboard GPS," *Advances in Astronautical Sciences*, Vol. 86, pp. 521–532; also AAS Paper 94-063.

[3] Axelrad, P., and Behre, C. P., "Attitude Estimation Algorithms for Spinning Satellites Using Global Positioning System Phase Data," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 1, 1997, pp. 164–169.

[4] Conway, A., Montgomery, P., Rock, S., Cannon, R., and Parkinson, B., "A New Motion-Based Algorithm for GPS Attitude Integer Resolution," *Navigation*, Vol. 43, No. 2, 1996, pp. 179–190.

[5] Garrick, J., "Investigation of Models and Estimation Techniques for GPS Attitude Determination," *1996 NASA-GSFC Flight Dynamics Division Flight Mechanics/Estimation Theory Symposium*, NASA CP-3333, NASA Goddard Space Flight Center, Greenbelt, MD, May 1996, pp. 89–98.

[6] Hwang, P. Y. C., "Kinematic GPS for Differential Position: Resolving Integer Ambiguities on the Fly," *Navigation*, Vol. 38, No. 1, 1991, pp. 1–15.

[7] Wahba, G., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 7, No. 3, 1965, p. 409.

[8] Shuster, M. D., and Oh, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, 1981, pp. 70–77.

[9] Markley, F. L., "Attitude Determination Using Vector Observations: A Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences*, Vol. 41, No. 2, 1993, pp. 261–280.

[10] Bar-Itzhack, I. Y., "REQUEST: A Recursive Quest Algorithm for Sequential Attitude Determination," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 5, 1995, pp. 1034–1038.

[11] Black, H. D., "A Passive System for Determining the Attitude of a Satellite," *AIAA Journal*, Vol. 2, No. 7, 1964, pp. 1350, 1351.

[12] Bar-Itzhack, I. Y., and Harman, R. R., "Optimized TRIAD Algorithm for Attitude Determination," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 1, 1997, pp. 208–211.

[13] Wertz, J. R. (ed.), *Spacecraft Attitude Determination and Control*, Reidel, Dordrecht, The Netherlands, 1984, p. 764.

[14] Leithmann, G. (ed.), *Optimization Techniques with Application to Aerospace Systems*, Academic, New York, 1962, pp. 210–212.